The Accuracy of ‘Power’ Maps to Display Curvature Data in Corneal Topography Systems

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**Purpose.** To quantify the error introduced by videokeratographic corneal topography devices in using a paraxial formula to calculate power over the entire corneal surface, including areas removed from the central paraxial region where the formula is known to be invalid.

**Methods.** Corneal refractive power and two paraxial power approximations were computed as a function of distance from the apex for three theoretical surfaces, a sphere and two ellipsoids with 0.3 and 0.5 eccentricities. Color dioptric maps were then theoretically created.

**Results.** For the spherical surface, both curvature-based paraxial power approximations were uniform over the entire surface because curvature is constant. However, the corneal refractive power increased from center to periphery, demonstrating the known phenomenon of spherical aberration. For the ellipsoids, which have been shown to model the human cornea, curvature-based power approximations decreased from center to periphery because the curvature flattens peripherally. However, refractive power increased from center to periphery. The limits of the central paraxial region for these surfaces was shown to be approximately 2 mm in diameter for the paraxial power approximation used by videokeratographic devices to measure 8 mm in diameter.

**Conclusions.** The direct correlation between corneal curvature and power with which clinicians are familiar is not valid in the peripheral regions measured by videokeratographic devices. Topographic devices measure curvature, which should not be interpreted as corneal power except in the central region. A recommendation to device manufacturers is to display "color curvature maps" instead of color dioptric maps, and to label the color bar with curvature values instead of power. Invest Ophthalmol Vis Sci. 1994;35:3525–3532.

Videokeratographic devices for measuring corneal topography are in widespread clinical use for a variety of purposes, including diagnosis and monitoring of keratoconus, planning and evaluating corneal refractive surgeries, fitting contact lenses, and others. The devices are based on Placido ring technology and measure curvature as a function of ring location along meridians in an image of concentric rings that has been reflected from the corneal surface. In 1987, the color-coded map was introduced for data display. It was thought that a two-dimensional color map would provide a tool that clinicians would find easy to interpret, rather than some sort of three-dimensional or numeric representation of the data. The idea was adopted by the industry, and current instruments are all equipped with similar display features in which corneal surface variations are represented as variations in color on the output map. However, instead of labeling areas of similar color with the corresponding curvature value, the curvature data are converted to "local power" by applying a paraxial power formula to the entire corneal surface.

Thus, the output of videokeratographic devices is referred to as a color dioptric map and is labeled in diopters as the name implies. This was done to provide the data to clinicians in terms with which they were familiar, namely diopters. However, before the development of corneal topographic devices, only the central cornea was measured, an area that may be considered paraxial. In the paraxial region, power and curvature are directly proportional, and it is appropriate to consider the terms as interchangeable. In other words, greater curvature corresponds to greater focal power,
and vice versa, in the central region. This is a concept that is well known and understood by clinicians. The relationship, however, is not valid in the peripheral regions that are measured by videokeratographic devices. By labeling the paraxial and nonparaxial areas using the same power formula, the impression is created that power and curvature are interchangeable over the entire corneal surface rather than being limited to the center. In fact, in the conventional terminology that has been proposed for corneal topography, it is explicitly stated that steeper areas have greater refractive power, flatter areas have less refractive power, with the assumption that areas of uniform radius of curvature correspond to areas of uniform refractive power.

Both curvature and power can provide important, yet distinctly different, types of corneal information to the clinician and vision scientist. For example, interpreting the videokeratographic data as curvature has proven useful clinically in evaluating the shape and symmetry of corrections produced by refractive surgeries. Power, on the other hand, provides information about the image forming properties of the cornea. Based on the current assumption that areas of uniform curvature correspond to areas of uniform refractive power, the videokeratographic data are also interpreted clinically and reported in the literature as power.

This is reinforced by the technical manuals that accompany the devices. The EyeSys Corneal Analysis System manual describes the same dataset interchangeably as either radius of curvature in millimeters or power in diopters, and the user may toggle between the two on the color bar scale.

Four distinct definitions of corneal "power" have been reported in the vision literature, as summarized by both Mandell and Klein. The four definitions present some confusion as to what the term power actually means. The issue may be clarified by recognizing that two of the definitions have a sound theoretical base in optics, and two are simply paraxial approximations that have been applied outside the paraxial region. The two definitions based in optics are functions of focal length and, thus, reflect focal power. First is the posterior focal power, based on the secondary focal point for incoming parallel rays. Mandell, Klein, and this investigator, all agree that this is the correct power definition for the eye as an optical imaging system. This definition will be referred to as corneal refractive power, $P_r$, in this article and represents the actual focal power of the cornea as a refractive element. Second, the anterior focal power has also been described based on the primary focal point for outgoing parallel rays. However, this is not appropriate for the eye in which image formation occurs in only one direction for rays entering the eye, not leaving it. Therefore, this definition will not be addressed in this article. The remaining two definitions are functions of curvature rather than focal length. Both are based on paraxial formulas, which will be detailed in the Methods section, and neither can account for spherical aberration. One is a function of the instantaneous radius of curvature along a specified meridian. This definition will be referred to as paraxial power approximation #2, $P_{a2}$, in this article. The last definition approximates radius of curvature with the distance from the surface point to the optical axis along the normal. In other words, the center of curvature is placed directly on the optical axis, which is accurate only for spherical surfaces.

It has been recognized that the paraxial power formula used by videokeratographic devices is not valid outside the central corneal region, but the magnitude of the error and the potential for misinterpretation of the corneal image-forming properties have not been thoroughly examined. Therefore, the purpose of this paper is to investigate the accuracy of the paraxial assumption and to quantify the error for two well-defined surface geometries: spherical and ellipsoidal. The ellipsoidal surface has been described as an appropriate first-order model for the human cornea.

**MATERIALS AND METHODS**

The three power formulations (corneal refractive power, and the two paraxial approximations) were investigated using three theoretical surfaces: a sphere of apical radius 7.5 mm, and two ellipsoids of apical radius 7.5 mm, with eccentricities of 0.3 and 0.5. All surfaces represent conic sections and are described as a rotation of the following equation about the $z$-axis:

$$x = \sqrt{(2R_0z) + (e^2 - 1)z^2}$$

where $R_0$ is the apical radius of curvature, $e$ is the eccentricity, the $z$-axis is equivalent to the optical axis, and $x$ represents the lateral distance from the optical axis. Equation (1) describes a conic section with its vertex at $(0,0)$ and its center at $(R_0/(1-e^2),0)$. For the spherical surface, $e = 0$. A schematic diagram of the surface cross-section within the specified coordinate system is given in Figure 1. The proportions are exaggerated for illustrative purposes. In this diagram, the cornea is labeled as a single refracting surface. The curve has its vertex at $A$, which corresponds to $(0,0)$ of the coordinate system. An off-axis ray, parallel to the optical axis, strikes the curve at surface point, $S$,
and incident angle, \( \theta_i \), to the normal. The ray is transmitted and refracted at angle \( \theta_t \), causing it to cross the optical axis at the secondary focal point, \( F' \). The point \( C \) represents the center of curvature, and the point \( D \) represents the intersection of the normal with the optical axis. The quantity, \( d \), represents the axial distance from \( S \), along the normal, to \( D \) on the optical axis. The quantity, \( r \), is the instantaneous radius of curvature at surface point, \( S \). For a sphere, \( C \) and \( D \) will be superimposed, and \( d \) will equal \( r \). The distance, \( f' \), is the secondary focal length. The locations of \( S \), \( D \), \( C \), and \( F' \) for this curve are all a function of \( x \), the lateral distance of the ray from the optical axis.

For the purposes of comparison to existing devices, all power calculations are done with the conventional keratometric index of refraction of 1.3375, which approximates the cornea as a single refracting surface by accounting for the cornea–aqueous interface. It is recognized that if purely front surface power is desired, the corneal index of refraction, 1.376, must be used.

**Corneal Refractive Power (P<sub>R</sub>)**

The first power formulation to be described is corneal refractive power, \( P_R \). It is defined in terms of how a ray will refract at a point as a function of index of refraction and incident angle for incoming rays parallel to the optical axis. It is calculated using Snell’s law. Secondary focal length, \( f' \), of a point on the corneal surface is defined as the distance from the corneal vertex to the point of intersection of the refracted ray with the optical axis for an incident ray parallel with the optical axis and is given by the following formula first described by Gullstrand:

\[
d = \frac{SD}{r} = \frac{SC}{f'}
\]

**FIGURE 1.** Schematic diagram of the cornea as a single refracting surface within a specified coordinate system. Proportions are exaggerated for illustrative purposes.

\[
f' = z + x \cdot \cot(\theta_i - \theta_t)
\]

where \( \theta_i \) is the incident angle, \( \theta_t \) is the transmitted refracted angle, and the coordinate system is defined with the \( z \)-axis equivalent to the optical axis with the corneal vertex at \((0,0)\) and \( x \) representing the perpendicular distance of a surface point from the optical axis, as illustrated in Figure 1. For the single refractive surface model, corneal refractive power, \( P_R \), is inversely related to the focal length by the index of refraction. Substituting equation (2) for focal length in this relationship results in the following equation, also described by Klein:

\[
P_R = \frac{n'}{f'} = \frac{n'}{z + \frac{x}{\tan(\theta_i - \theta_t)}}
\]

where the index of refraction, \( n' = 1.3375 \), and \( \theta_i, \theta_t, z, \) and \( x \) are already defined in Figure 1. The incident angle is determined via geometry to be the following:

\[
\theta_i = \sin^{-1}\left(\frac{x}{d}\right)
\]

and \( \theta_t \) is determined by Snell’s Law:

\[
n' \cdot \sin \theta_t = \sin \theta_i
\]

**Paraxial Power Approximation #1 (P<sub>a1</sub>)**

In paraxial power approximations, the angle of incidence and the transmitted, refracted angle are small so that their sine and tangent may be approximated by the angle itself. In other words, \( \sin \theta_i = \theta_i \), \( \sin \theta_t = \theta_t \), and \( \tan(\theta_i - \theta_t) = \theta_i - \theta_t \). If these substitutions...
are made in equations (3), (4), and (5), and the results are combined along with the approximation that $z = 0$ in the paraxial region, equation (6) follows.

$$P_{sa} = \frac{(n' - 1)}{d} \tag{6}$$

where $P_{sa}$ refers to paraxial power approximation #1, $n' = 1.3375$, and $d$ is the distance from the surface point to the optical axis along the normal, as shown in Figure 1. Paraxial approximation #1 is known to model the behavior of topographic instruments.\textsuperscript{7,10} The distance, $d$, which varies with curvature, was calculated according to the following equation, described elsewhere,\textsuperscript{10} for the distance from surface to axis along the normal of a prolate ellipse:

$$d = \sqrt{(R_0^2 + x^2)}^{1.2} \tag{7}$$

**Paraxial Power Approximation #2 (P_{sa2})**

Paraxial power approximation #2, $P_{sa2}$, is also defined as a function of local curvature at a point. The equation is as follows:

$$P_{sa2} = \frac{(n' - 1)}{r} \tag{8}$$

where the index of refraction, $n' = 1.3375$, and $r$ is the radius of curvature of a surface point, as shown in Figure 1. Paraxial approximation #2 is identical to approximation #1 for a sphere, where $d = r$. This is actually the equation described by Klyce\textsuperscript{1} for topographic devices. However, it has since been shown that these devices do not accurately measure the instantaneous radius of curvature, but they more closely model the axial distance, $d$.\textsuperscript{10} The radius of curvature, $r$, was calculated according to the following equation, described elsewhere:\textsuperscript{10}

$$r = \frac{(R_0^2 + x^2)^{0.5}}{R_0^2} \tag{9}$$

where the variables have been previously defined.

For each of the three theoretical surfaces, power was calculated as a function of lateral distance from the optical axis using the definition of corneal refractive power and both paraxial power approximations. Color dioptic maps based on the three definitions were mathematically created and then displayed using graphic devices. However, it has since been shown that these devices do not accurately measure the instantaneous radius of curvature, but they more closely model the axial distance, $d$.\textsuperscript{10} The results are presented graphically in Figure 2. Each row of Figure 2 represents a different surface and each column a different power formulation. Therefore, all three maps in a given row correspond to the same surface, with one map for each power formulation. Also, each column represents a given power formulation, with one map for each surface. The left column of three maps represents the corneal refractive power for each surface. The middle column represents paraxial power approximation #1, which is used by corneal topography instruments, as previously verified by measuring corresponding ellipsoidal test surfaces.\textsuperscript{10} The far right column represents paraxial power approximation #2 for each surface, based on radius of curvature. The top row of maps corresponds to the sphere. For the sphere, $P_{a1} = P_{a2}$ because the radius of curvature equals the distance from the curve to the axis along the normal, or $r = d$. The two paraxial power approximations produce uniform maps, as expected for a sphere with constant curvature. However, the refractive power map demonstrates increasing power from center to periphery. This is the result of spherical aberration. The middle row of maps corresponds to an ellipsoid of eccentricity 0.3. Both paraxial approximations demonstrate similar patterns of decreasing power from center to periphery, with a greater decrease in approximation #2. The refractive power map, on the other hand, demonstrates increasing power from center to periphery. The bottom row represents an ellipsoid of eccentricity 0.5. The results are qualitatively similar to the 0.3 ellipsoid with a pattern of decreasing power for both paraxial approximations, with greater decrease in approximation #2. The refractive power map again demonstrates the opposite pattern of increasing power from center to periphery. These results are presented graphically in Figure 3.

**DISCUSSION**

Although it has been recognized that the paraxial power approximations will not be accurate outside the central corneal region,\textsuperscript{7,8} the potential error has not been quantified. The surprising results of this study indicate that not only is the error significant, the paraxial power approximations actually produce a completely opposite pattern from the corneal refractive power. In both ellipsoids, where the paraxial approximations show decreasing values, the corneal refractive power is actually increasing! Because the maps produced have different, opposing patterns, the results of the paraxial approximations may not even be interpreted qualitatively as power in a clinical environment. For example, in the 0.5 ellipsoid at $x = 2.25$ mm, the true refractive power is 45.66 D, paraxial approximation #1 is 44.50, and paraxial approximation #2 is 43.54 D, with paraxial errors of 1.16 D and 2.12 D, respectively. The paraxial error becomes even more profound, however, if one considers the difference
from the central value rather than just the absolute numeric values. The corneal refractive power increases 0.66 D from the apical value of 45 D, whereas paraxial approximation #1 decreases 0.5 D and paraxial approximation #2 decreases 1.46 D. This may have an extremely important impact in clinical interpretation because the human cornea is known to be approximately ellipsoidal in shape.11,12

Two major conclusions result from this study. First, areas of similar curvature do not correspond to areas of similar refractive power in the peripheral regions, as previously assumed. Second, paraxial power approximations are not valid outside the central corneal region and may produce erroneous patterns that could lead to clinical misinterpretation of the image-forming properties of the cornea. The limits of the paraxial region depend upon the geometry of the surface itself. Operationally, this may be defined as that region for which the corneal refractive power map and the paraxial power approximation maps have the same color. This is easy to see in Figure 2, where the paraxial region corresponds operationally to the central green area in all of the refractive power maps, indicating 45 D. For the sphere and the 0.3 ellipsoid under study, the operational limits of the paraxial region are at 0.75 mm from the apex for the color step size of 0.6 D that is shown in Figure 2. At this point, which is marked with the cursor, the first color change occurs and the powers diverge in direction. For the 0.5 ellipsoid, the operational limits of the paraxial region are slightly larger, at 1 mm from the apex at the first color change. This operational definition, however, will differ depending on the step size chosen on the color bar scale. Therefore, the videokeratographic paraxial region may be theoretically defined as that region in which the corneal refractive power and the paraxial power approximations differ by 0.25 diopters or less. With this definition, the paraxial regions for each of the theoretical surfaces analyzed in this study were calculated and are given in Table 1. It can be seen in Table 1 that for Pp1, the limit of the paraxial region is approximately 1 mm radial distance for all surfaces, or the central 2 mm diameter of the cornea. On the other hand, Pp2 shows consistently smaller paraxial regions.

The reason for the inconsistent results between corneal refractive power and the paraxial power approximations is illustrated in Figure 4 for the sphere and the 0.5 ellipsoid. In the paraxial formulations, given in equations (6) and (8), power is a function only of index of refraction and curvature. The crucial quantity not considered is the angle of incidence. In the paraxial region, the angle of incidence is small, and the sine and tangent of the angle may be approximated by the angle itself, as previously described. However, the further from the optical axis a parallel ray strikes the surface, the larger the angle of incidence, and the sine and tangent approximations can no longer be made. The larger the angle of incidence, the larger the transmitted, refracted angle and the greater the focal power. Figure 4A illustrates this concept for a spherical single refracting surface. Light rays strike the two surfaces points, S1 at x = 1 mm and S2 at x = 4 mm. The radius of curvature, r, and the axial distance, d, are identical for both points. Therefore, paraxial approximations would assign the same power to both. However, because the angle of incidence is larger for S2, the corresponding refracted ray crosses the optical axis 2.37 mm closer to the vertex than that of S1. Figure 4B illustrates the concept for an ellipsoidal single refracting surface, with eccentricity of 0.5. For this surface, both the radius of curvature and the axial distance are longer for S2 than for S1. Therefore, the paraxial formulas would both show decreasing power for S2. However, the angle of incidence at S2 is still larger than at S1, and the refracted ray at S2 still crosses the optical axis closer to the vertex than that of S1. Therefore, the corneal refractive power is greater at S2 than at S1, which is opposite to the paraxial formulations.

For the topographic device industry, it is the recommendation of this investigator to change the "color dioptric power maps" to "color curvature maps," and to label the color bar scale with curvature values only. In other words, there would be no change in the actual output map itself, in terms of color or pattern, only in labeling the display. This would be an easy change to make with a simple software update. The device manuals would have to be updated to reflect the change and remove references to power. It would then be a valuable addition to develop new algorithms to compute corneal refractive power based on the topo-

FIGURE 2. Color coding indicates corneal areas of similar power, with warmer colors indicating higher power and cooler colors indicating lower power according to the scale at the top. Boxes list information for the point under the cursor, including radius of curvature, power, meridian, and distance from apex. All surfaces have apical radius of 7.5 mm. Top row: Sphere. Middle row: Ellipsoid with 0.3 eccentricity. Bottom row: Ellipsoid with 0.5 eccentricity. Left column: Corneal refractive power. Middle column: Paraxial power approximation #1. Right column: Paraxial power approximation #2. (Note: Radius of curvature is not included in the first column maps because it is not proportional to power in this formulation.)
graphic data obtained. This could be used to complement the curvature maps for clinicians and vision scientists. Thus, two maps would be computed for each surface. Refer once again to Figure 2, but instead of interpreting the far right column as inaccurate paraxial power approximations, interpret it as accurate curvature maps. It is then readily understood that a sphere has constant curvature and, thus, a uniform curvature map. However, it has increasing optical power because of spherical aberration, which is displayed in the refractive power map in the left column. Both ellipsoids demonstrate decreasing curvature peripherally, accurately reflected in the far right column maps, but increasing refractive power, accurately reflected in the left column maps. Though the maps in the middle column of Figure 2 are qualitatively similar in pattern to those in the right column, it has been previously shown that quantitatively, the data do not correspond to radius of curvature but to axial distance. Although axial distance is related to curvature, it is not inversely proportional, which is the relationship for radius of curvature. Therefore, the middle column maps may be interpreted qualitatively as curvature maps, but not quantitatively.

If clinicians and vision scientists are presented with both power and curvature information, it would

### Table 1. Radial Distance of the Paraxial Region* for Each Paraxial Formulation for Surfaces With Apical Radius of Curvature = 7.5 mm

<table>
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<tr>
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<th>$P_{a1}$ (mm)</th>
<th>$P_{a2}$ (mm)</th>
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<tbody>
<tr>
<td>Sphere</td>
<td>1.05</td>
<td>1.05</td>
</tr>
<tr>
<td>0.3f Ellipsoid</td>
<td>1.05</td>
<td>0.92</td>
</tr>
<tr>
<td>0.5f Ellipsoid</td>
<td>1.06</td>
<td>0.77</td>
</tr>
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* Paraxial region defined where corneal refractive power minus paraxial power approximation is less than or equal to 0.25 diopters.

† Eccentricity of the ellipsoid.
be possible to choose the appropriate map or maps necessary for the specific application. For example, in radial keratotomy, the curvature map would indicate to the surgeon the symmetry and shape of the induced correction. The surgeon might infer that the steepening periphery would cause some potential visual disturbance but would not be able to quantify it. On the other hand, the power map would indicate much larger amounts of spherical aberration because the steepening periphery would make the angle of incidence for parallel rays even larger than demonstrated in this article for flattening peripheries. This quantitative information might aid in understanding the visual distortions experienced by patients with radial keratotomy at night when their pupils dilate to allow light refracted in the peripheral region to reach the retina.

The focal power calculations would also be useful to those researchers using topographic data for developing corneal raytracing algorithms. Currently, the dioptric power output of the videokeratographic device is interpreted as focal power to predict image information based on object information. The results of the current study indicate that using topographic data in this way will not produce accurate image formation results, except for the central cornea. It is first necessary to reconstruct the unknown surface to calculate the refractive properties of that surface in an imaging situation. The results of the current study will, it is hoped, prevent misinterpretation of curvature data as focal power data and lead researchers into developing surface reconstruction algorithms based on videokeratographic data that would then lead to accurate refractive power algorithms.

The ultimate implications of this study are that curvature should be displayed and interpreted as curvature or related to shape, and optical power should be displayed and interpreted as optical power or related to the focal properties. They are complementary, not interchangeable, except in the central paraxial region of the cornea. Therefore, it is crucial that videokeratographic data, which covers most of the corneal surface, be presented in appropriate form to ensure accurate interpretation by clinicians and vision scientists. Curvature and optical power should be considered as separate, but equally meaningful, quantities. Both are important in understanding the relationship between topographic shape and the image-forming properties of the cornea.

**Key Words**
corneal topography, videokeratoscopy, refractive power, sphere, accuracy

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**References**


