Perimetry: Principal Component Varimax Rotation Followed by Validated Cluster Analysis

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PURPOSE. To extract unidimensional, well-separated latent scores that are anatomically and clinically valid from 52 standardized variables collected by Humphrey visual field (VF) perimetry (Carl Zeiss Meditec, Dublin, CA).

METHODS. Visual field data of 437 patients were collected and classified by a glaucoma specialist into seven clinical groups: irregularities of VF (IVF), nasal step (NaS), arcuate scotoma (AC), paracentral scotoma (PCS), blind-spot enlargement (BSE), diffuse deficit (DD), and advanced deficit (AD). The number and content of constituent variable scores were identified by principal components analysis followed by Varimax Rotation and simple clustering, taking spatial distribution homogeneity and visual system anatomy into account. Unidimensionality was checked by a stepwise Cronbach α curve. Clinical predictability of the derived scores was checked by comparing clinical groups (ANOVA).

RESULTS. Patients older than 60 years comprised 53.3% of the sample. The average mean deviation was −9.2 dB and pattern standard deviation was 6.5 dB. Six scores were identified: four peripheral scores (nasal superior, NS; nasal inferior, NI; temporal superior, TS; and temporal inferior, TI) and two paracentral scores (PCSs; superior, PCSS; and inferior, PCSI). Cronbach α was always >0.90. The six scores decreased sequentially from IVF to DD to AD. Scores of AC were lower in NS, NI, and TS; PCSS was less in PCS; BSE scores were less in TS and TI; NaS scores were less in NS and NI.

CONCLUSIONS. Six well-separated, optimal scores were obtained from the Humphrey perimetry matrix. Internal reliability was good. It was possible to discriminate between clinical subgroups. Further analyses, based on longitudinal data, must be performed to confirm these findings. (Invest Ophthalmol Vis Sci. 2005;46:3169–3176) DOI:10.1167/iovs.04-1214

Automated static perimetry is one of the methods used to screen and follow up patients who have glaucoma.1–4 It consists of approximately 100 quantitative threshold measures that permit evaluation of retinal sensitivity. Each measure is standardized in a population free of ocular disease, and two simple statistics are calculated: mean deviation (MD) and pattern standard deviation (PSD). These indices are widely used in glaucoma clinical trials and patient follow-up.

Mean deviation (MD) is the average of all differences between measures and their normal values, weighted by the variance observed in the general population (where $X_i$ is the measured threshold, $N_i$ is the normal reference threshold at point $i$, $S_i$ is the variance of normal field measurement at point $i$, and $n$ is the number of test points).

$$\text{MD} = \frac{1}{n} \sum_{i=1}^{n} \frac{(X_i - N_i)}{S_i}$$

PSD is a normalized distance, standardized with reference to the general population, calculated for each point.

$$\text{PSD} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(\frac{(X_i - N_i - \text{MD})^2}{S_i} \right)}$$

Although there is a considerable body of literature about the properties of these indices as they relate to the clinical picture, some weaknesses can be identified a priori: (1) They do not take into account the spatial distribution of the points measured, the proximity of one point to another, and correlations between the points (e.g., switching one measure with another changes neither the MD nor PSD); (2) MD cannot be interpreted without knowing PSD, and vice versa; (3) MD is not a sensitive parameter in early stages of glaucoma; (4) PSD is not a sensitive parameter in late stages of glaucoma; (5) neither measure takes into account anatomic dimensions of the eye or visual system (horizontal threshold of retinal nerve fibers, the vertical threshold of vision cerebral hemispheres, and the retinal artery located centrally in the optic nerve).

Very little research has been performed to find algorithms that would help to identify visual field defects (VFDs) more precisely. Brigatti et al.6,7 used computerized neural networks with some success to identify patients with early glaucomatous visual field loss, yielding sensitivity and specificity both >70%. To achieve this, they had to include information on the automated visual field index and other structural data.

Mandava et al.8 identified 11 clusters by nearest-neighbor cluster analysis performed on Octopus visual fields (Haag-Streit, Küniz, Switzerland). A discriminant analysis was performed on the 11 scores used to classify patients with and without glaucoma. The sensitivity and specificity of this classification were very good (sensitivity and specificity >90%).

Brigatti et al.7 and Mandava et al.8 shared a common objective to develop a classifying algorithm that would help clinicians to detect new VFDs. Brigatti et al.7 directly included global perimeter indicators that assume that MD, PSD, and short-term fluctuation are the optimal information that can be retrieved from this test. However, the model produces estima-
tors that are completely disconnected from clinical reality, and clinicians still must be persuaded by the conclusions of neural network modeling. Mandava et al. used a cluster analysis with the sole purpose of optimizing information reduction before running a discriminant analysis. However, the clusters did not respect retinal anatomy very well, for example, they spanned the vertical meridian, and almost nothing was stated about either the construct validity or the internal consistency reliability of these clusters.

Although we agree with the approach developed by Mandava et al., we believe that the scores developed should be demonstrated to be clinically relevant before they are applied in the identification and monitoring of patients with glaucoma by using specific statistical techniques—for example, neural networks or discriminant function.

The present pilot study was designed to identify scores produced by automated static perimetry that would: (1) be structurally valid, (2) respect the anatomy of the eye and visual system, (3) follow the clinical evolution of glaucoma, and (4) be easily interpreted by physicians. This article addresses only the development and validation of the scores.

**MATERIALS AND METHODS**

**Data Collection**

A cross-sectional experimental design was used. All patients were recruited from the glaucoma department of the Hôpital des Quinze-Vingts. Consecutive perimetry data from one eye (left or right) of each patient were selected from the medical records. Patients were of either sex with no age limit and were required to have medically confirmed glaucoma, presenting with at least one VFD. Patients with no VFD were not included because they did not contribute valuable information to the correlation matrix. As a general rule, any type of glaucoma was accepted (congenital, inflammatory, or neovascular, with narrow or closed angle, after cataract surgery), since all types contribute valuable information. Based on the available data, all patients had primary open-angle glaucoma. This research adhered to the tenets of the Declaration of Helsinki and also observed the guidelines in French law regarding data privacy.

Both eyes were taken into account when glaucoma was bilateral. To pool data from all eyes, a left mirror image of right eyes was computerized. A potential laterality effect on scores was checked by an analysis of variance (ANOVA).

All data used had been obtained according to the usual procedure of the glaucoma department. Patients, by default therefore, underwent a Humphrey perimeter 24-2 threshold test, with the FASTPAC algorithm (Carl Zeiss Meditec, Dublin, CA). The main characteristics of FASTPAC is the use of 3-dB steps. The fixation target was centralized on the blind spot used to monitor fixation. The stimulus was size III, white, and on a background of 31.5 asb. Patients were required to have experienced a previous visual field test. Only tests with normal reliability indices were included: fixation losses <20% and false-positive and -negative errors <30%.

On the basis of these data, a glaucoma specialist clinically classified patients into seven groups, according to VFD shape: (1) irregularities of visual field; (2) nasal step; (3) arcuate scotoma; (4) paracentral scotoma; (5) blind spot enlargement; (6) diffuse deficit; and (7) advanced deficit. Visual field irregularities corresponded to slightly depressed thresholds throughout the visual field, which did not amount to specific focal defects. Diffuse defects corresponded to a generalized loss of sensitivity throughout the whole field. Advanced glaucoma is defined by multiple scotoma surrounding the fixation point.

Data were entered into the study database with double-entry control. Analysis was conducted on the pattern deviation matrix since: (1) the data are standardized against a normal population; and (2) it reveals localized defects that may be masked by a generalized depression or an elevation of the hill of vision.

![Figure 1. Variable names used for the clustering analysis.](image)

Control patients without glaucoma were not included because the pattern deviation matrix is already standardized according to an age- and gender-matched normal population. In addition, the Varimax Rotation, used for optimization of the data, would have been affected by the presence of two (control and patient) subjects with different patterns of correlation.

Figure 1 describes the variable labels used in the analysis. Fifty-two variables were identified from the threshold data, as follows: 14 in the northwest quadrant (NW1–NW14); 12 in the northeast quadrant (NE1–NE11, NE13); 14 in the southwest quadrant (SW1–SW14); and 12 in the southeast quadrant (SE1, SE2, SE4–SE13). This nontraditional terminology is used to facilitate understanding of the Varimax Rotation used and the resultant clusters. MD and PSD were collected as the main parameters from the perimeter test.

**Clustering of Variables and Construction of Scores**

The statistical analysis was conducted on computer (SAS software, ver. 8.2; SAS Institute, Cary, NC). Two-sided tests were performed with a = 5%. Our sample size was fixed according to empiric rules used in factor analysis—that is, 5 to 10 patients should be included per variable in the analysis; hence, >400 patients were included for the 52 variables identified.

A principal components analysis (PCA) was performed on the 52 variables to retain all meaningful information. The number of factors retained was chosen after examining the plot of eigenvalues. Only factors with eigenvalues >1 were retained. Scores were extracted after Varimax Rotation followed by a cluster analysis with the number of clusters fixed and equal to the number of factors retained—that is, each of the 52 variables was classified into the most correlated rotated factor. An unweighted score was computed for each factor by summing the variables that correlated with it the most. Each variable was included in one score only. This clustering step was followed by a refining process, described later, that took into account the specificity of the data.

Correlations between each of the 52 variables and the scores were computed. A variable was regarded as correctly classified when its correlation with a score was higher than that with any other score.

When this was not the case, the variable was moved to other scores to maximize its correlation. This iterative process continued until the system stabilized (i.e., no further movement was necessary). Some specific rules were added regarding the spatial homogeneity and clinical coherence of the clusters: (1) To maintain the spatial homogeneity of clusters (geographical continuity: a score had to relate to neighboring measurements), constraints were defined that restricted certain moves—that is, if a measurement was separated spatially from its cluster by another measurement, the former was moved to the nearest cluster. (2) To maintain the clinical coherence of clusters, specific constraints were defined that restricted certain moves—that is, if a variable was in a cluster not compatible with known medical constraints underlying eye function, it was moved to the nearest, clinically meaningful cluster. This rule applied when a cluster crossed the east-west or north–south axes.
Assessment of Unidimensionality and Clinical Validity of Scores

A stepwise Cronbach α curve\(^2\) was plotted to check the unidimensionality of the variables yielding the score. This calculation made it possible to verify that a group of items measured the same underlying unidimensional concept (construct validity). The curvature should increase monotonically when all items belong to the appropriate score. Otherwise, items should be allocated to another score. However, because it is influenced by sample fluctuations, it should be interpreted cautiously, especially when scores contain few items.

Clinical validity can be estimated as the ability of a score to capture clinical relevance.\(^1\) The mean of each score was compared across all seven clinical groups by ANOVA.

RESULTS

In total, 437 consecutive visual fields were collected of which 229 (52.4%) were right eyes. Mean age was 61.2 ± 14.8 years (SD) and 53.3% of patients were older than 60. Patients manifested the following visual field abnormalities: irregularities of visual field (\(n = 34\); 7.8%); nasal steps (\(n = 126\); 28.8%); arcuate scotomas (\(n = 154\); 35.2%); paracentral scotomas (\(n = 21\); 4.8%); blind-spot enlargements (\(n = 13\); 3.0%); diffuse deficits (\(n = 26\); 5.9%); and advanced deficits (\(n = 63\); 14.5%).

Pupil diameter was documented in 137 patients and, on average, was 4.3 mm ± 1.1 (SD). The average duration of the test was 8.0 minutes. The average MD was −9.2 ± 7.2 dB (SD) and the PSD was 6.5 ± 3.3.

Principal component analysis identified six factors that explained 61.09% of the total variance. After an abrupt decrease, the plot of the eigenvalues showed a clear break at the sixth eigenvalue, then a plateau, and again a new, but slighter, decrease. The sixth eigenvalue was also the first value less than unity,\(^1\) and so the min-eigen criterion also retained six factors.

Figure 2 describes step-by-step how the scores were constructed. Examination of correlations with the six factors retained after Varimax Rotation indicated the first cluster pattern of the original variables. All measurements of the northwest quadrant plus NE3, NE6, and NE11 were correlated and constituted factor 1 (correlations: 0.53–0.85). Factor 2 comprised all measurements of the southwest sector (correlations: 0.59–0.85). Factor 3 included all measurements of the northeast sector (correlations: 0.40–0.75) except NE3, NE6, and NE11, already attracted by factor 1, and NE10, which alone represented factor 6 (correlation: 0.49). Factor 4 comprised measurements of the southeast sector (correlations: 0.54–0.70) except SE5 and SE6, which jointly made factor 5 (correlations: 0.53 and 0.65).

The six rotated factors gave us an initial set from which to build six simple scores. When examining empiric correlations between the original variables and these scores, we found that NE3 correlated more with score 3 (from factor 3) than with its own score 1 (from factor 1). Similarly, NE11 correlated more with score 6 than with its own score 1, and SW5 correlated more with score 5 than with its own score of 2 (Fig. 2: scores after the first correlation analysis). Moves were performed accordingly.

Correlations were calculated again, and we were obliged to move NE6 from score 1 to score 3, NW14 to score 6, and both SE1 and SE2 from score 4 to score 5. (Fig. 2: scores after the second correlation analysis). No additional moves were necessary after the second step.

Cronbach α curves were estimated for each subset to check that the scores were unidimensional. We were obliged to move NE13 and SE4 to another set to obtain unidimensional scores. If NE13 was moved to score 4, we produced a set crossing the north-to-south quadrants. Moreover, the new set (score 4 + NE13) was no longer unidimensional. A similar situation arose when we tried to move SE4 to score 3. NE13 and SE4 were therefore not moved. NE6 and NE7 could, however, be moved to score 6, in accordance with the adopted rules. In this way, spatial homogeneity was respected (no crossing from the north to the south quadrant), as was specificity (each variable correlated more with its own score than with any other score).

The six clusters from which scores were derived are described in Figure 2. Four scores were peripheral (nasal superior, NS; nasal inferior, NI; temporal superior, TS; temporal inferior, TI) and the remaining two central (paracentral superior, PCS; and paracentral inferior, PCI). The formulas were respectively:

\[
\text{NS} = \frac{(\text{NW1} + \text{NW2} + \text{NW3} + \text{NW4} + \text{NW5} + \text{NW6} + \text{NW7} + \text{NW8} + \text{NW9} + \text{NW10} + \text{NW11} + \text{NW12} + \text{NW13})}{13}
\]

\[
\text{NI} = \frac{(\text{SW1} + \text{SW2} + \text{SW3} + \text{SW4} + \text{SW5} + \text{SW6} + \text{SW7} + \text{SW8} + \text{SW9} + \text{SW10} + \text{SW11} + \text{SW12} + \text{SW13} + \text{SW14})}{13}
\]

\[
\text{TS} = \frac{(\text{NE1} + \text{NE2} + \text{NE3} + \text{NE4} + \text{NE5} + \text{NE8} + \text{NE9} + \text{NE13})}{8}
\]
For each score, Cronbach $\alpha$ was always $>0.90$ (Fig. 3, the maximum value reported in the curve), demonstrating good internal reliability. Cronbach $\alpha$ curves for peripheral scores increased monotonically, supporting the requirement that all items should contribute to the score. However, this was not the case for central scores, although Cronbach $\alpha$ curves were always $>0.90$. Last, the decrease of curvature was small and limited to a single item.

Table 1 describes cross-correlations between the scores. Six of fifteen coefficients were $>0.70$ and most were between central and peripheral scores.

Table 2 and Figures 4 and 5 illustrate the six scores and the MD according to the type of clinical abnormality. As shown by the MD, all scores demonstrated highly significant differences across the seven groups of patients. At the extremes, patients
with advanced deficits produced lower MDs than those with VF irregularities. The MDs in Figures 4 and 5 were weighted averages of the six scores according to the size of the respective clinical groups. The scores showed significant variations around MD that were specific to clinical abnormalities. Patients with a nasal step had nasal superior and nasal inferior scores less than the MD, whereas the other four scores were greater than the MD. Patients with VFDs, or a diffuse deficit, did not demonstrate large differences between the six scores and the MD. Patients with an arcuate scotoma had nasal superior and temporal superior scores less than the MD, whereas the paracentral score was greater than the MD. Patients with a paracentral scotoma had temporal superior and temporal inferior scores greater than the MD, whereas the paracentral score was less than the MD. Patients with blind spot enlargement had temporal superior and temporal inferior scores less than the MD. Last, some variation was observed in even the most severe patients (advanced), as follows: the paracentral inferior and temporal inferior scores were greater than the MD, whereas the two nasal scores (superior and inferior) were less than the MD.

**Discussion**

MD and PSD are often used in randomized clinical trials as primary end points for the evaluation of glaucoma treatments. A low MD or a high PSD is associated with moderate or advanced stages of glaucoma. Consequently, both parameters are often used in randomized clinical trials as a low MD or a high PSD is associated with moderate or advanced stages of glaucoma. Consequently, both parameters are often used in randomized clinical trials as the MD. Patients with blind spot enlargement had nasal superior and nasal inferior scores greater than the MD, whereas the other four scores were specific to clinical abnormalities. Patients with arcuate scotoma had temporal superior and temporal inferior scores greater than the MD, whereas the paracentral score was less than the MD. Patients with an arcuate scotoma had nasal superior and nasal inferior scores less than the MD, whereas the paracentral score was greater than the MD. Patients with an arcuate scotoma had nasal superior and nasal inferior scores less than the MD, whereas the paracentral score was greater than the MD. Patients with an arcuate scotoma had nasal superior and nasal inferior scores less than the MD, whereas the paracentral score was greater than the MD. Patients with an arcuate scotoma had nasal superior and nasal inferior scores less than the MD, whereas the paracentral score was greater than the MD.

A PCA followed by our clustering algorithm allowed us to reach this objective. We identified six scores that explained more than 60% of the observed variance. The scores had orthogonal properties meaning that their independence was maximized. They had also good construct validity, as demonstrated by Cronbach $\alpha$ curves—that is, with the four peripheral scores, at least; switching an item to another score did not improve its validity. In the case of the two paracentral scores, some switching did improve reliability. However, the loss of reliability, as measured by the Cronbach $\alpha$ curve, was very low and could be explained by stochastic sampling issues. We therefore preferred to keep scores close to retinal anatomy (proximity of points), instead of maximizing the mathematical properties of our scores. Finally, the scores were easy to calculate.

We used the pattern deviation matrix and performed statistical manipulations to achieve this result. We could have worked on threshold sensitivities or the total deviation matrix. The former would have required an adjustment for age. The latter provides an indirect standardization, based on a population-wise approach, which is better than local data-based adjustments. We used the pattern-deviation matrix because it emphasized localized defects and therefore would increase the correlation between points belonging to a same VFD. This correlation should stabilize the Varimax Rotation.

Although we used a rotation pattern that maximized score independence, we still found high correlations between certain scores. This indirectly, but strongly, supports the fact that the MD is a score with a high construct validity. In other words, each item contributes homogenously to the MD. In contrast, a possible use of MD as a single score would explain a much smaller part of the total variance; hence, much information would be lost.

**Table 1. Correlation Coefficients between Scores**

<table>
<thead>
<tr>
<th>Cluster</th>
<th>Nasal Superior</th>
<th>Nasal Inferior</th>
<th>Temporal Superior</th>
<th>Temporal Inferior</th>
<th>Paracentral Superior</th>
<th>Paracentral Inferior</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nasal superior</td>
<td>1.00</td>
<td></td>
<td>0.58</td>
<td>0.80</td>
<td>0.53</td>
<td>0.45</td>
</tr>
<tr>
<td>Nasal inferior</td>
<td>0.58</td>
<td>1.00</td>
<td>0.53</td>
<td>0.79</td>
<td>0.76</td>
<td>0.76</td>
</tr>
<tr>
<td>Temporal superior</td>
<td>0.80</td>
<td>0.53</td>
<td>1.00</td>
<td>0.65</td>
<td>0.52</td>
<td>0.77</td>
</tr>
<tr>
<td>Temporal inferior</td>
<td>0.53</td>
<td>0.79</td>
<td>0.65</td>
<td>1.00</td>
<td>0.53</td>
<td>1.00</td>
</tr>
<tr>
<td>Paracentral superior</td>
<td>0.45</td>
<td>0.76</td>
<td>0.52</td>
<td>0.77</td>
<td>0.56</td>
<td>1.00</td>
</tr>
<tr>
<td>Paracentral inferior</td>
<td>0.80</td>
<td>0.76</td>
<td>0.78</td>
<td>1.00</td>
<td>0.56</td>
<td>1.00</td>
</tr>
</tbody>
</table>

All correlation coefficients were significantly different from zero ($P < 0.001$). According to Chassany et al., $^{15}$ a correlation coefficient between 0.4 and 0.7 is a guarantee of no redundancy.

Probability calculated according to ANOVA. Average scores by clinical groups were found to be statistically significant at $P < 0.0001$. Data are expressed as the mean $\pm$ SD.
Our algorithm was successful in producing scores that respected retinal anatomy (good and rapid convergence). Four quadrants, fully separated by the horizontal and vertical axes, were shown to be associated with two central scores. The coexistence of central and peripheral central scores could be interpreted as follows: Central scores may be sensitive to blood flow variations in the central retinal artery, whereas peripheral scores may be more sensitive to intraocular pressure’s effects on nerve fibers at the disc junction. Additional data are needed to confirm this hypothesis.

The relationship between the localization of a clinical field defect and its corresponding score demonstrated that our scoring algorithm was clinically relevant and thereby possessed construct validity. Apart from visual field irregularities and diffuse deficit, our six scores assembled valuable data describing the localization of a VFD.

Nasal scores were involved in nasal step and arcuate scotoma, temporal and paracentral scores in blind-spot enlargement, and paracentral scores in paracentral scotoma. Even with advanced deficits, the six scores added information to the MD. The relationship between the localization of a clinical VFD and its corresponding score demonstrated that our scoring algorithm was clinically relevant and thereby possessed construct validity.

Our six scores differed from the eleven described by Mandava et al. because of the algorithm used. We believe that fewer scores for retinal anatomy would be easier both to apply and understand in daily practice. Because our scores demonstrated good construct and external validity, they should assist in patient follow-up, although additional longitudinal data are needed to confirm this. Comparison with the work of Brigatti et al. is not straightforward, since their main goal was to identify patients with glaucoma. Nonetheless, our scores could be used as entry parameters in a neural network to serve the same purpose.

AGIS (Advanced Glaucoma Intervention Study) scores were not calculated for our sample of patients. Therefore, a head-to-head comparison with our six scores was not possible. The AGIS investigators decided that one single clinical score was appropriate to define the severity of glaucomatous VFDs. This assumption was somewhat contradicted by our findings. Our algorithm is also simpler than that used in AGIS and can be managed with a basic calculator. Finally, we demonstrated that our six scores brought additional information to the MD.

Our pilot study has limitations. A larger sample size may have increased the sensitivity of our algorithm to detect different or additional scores. The sample size of some patient groups was rather small, and certain analyses should be inter-
A Parallel Model Describing the Unidimensionality of a Set of Variables. Let $X_1$, $X_2$, ..., $X_n$ be a set of observed variables measuring the same underlying unidimensional latent (unobserved) variable. We define $X_{ij}$ as the measurement of patient $i$, where $i = 1, \ldots, n$, given by a variable $j$, where $j = 1, \ldots, k$. The model underlying Cronbach $\alpha$ is a simple, mixed, one-way model: $X_{ij} = \mu_j + \alpha_i + \varepsilon_{ij}$ where $\mu_j$ is a variable fixed (nonrandom) effect and $\alpha_i$ is a random effect with zero mean and $SE\sigma_\alpha$, corresponding to patient variability. It produces the variance of the true latent measure ($\tau_{ij} = \mu_j + \alpha_i$); and $\varepsilon_{ij}$ is a random effect with zero mean and $SE\sigma$ corresponding to the additional measurement error. The true measure and the error are uncorrelated: $cov(\alpha_i, \varepsilon_{ij}) = 0$.

These assumptions are classic in experimental design. This model defines relationships between different kinds of variables: the observed score $X_{ij}$, the true score $\tilde{X}_{ij}$, and the error $\varepsilon_{ij}$.

Reliability of an Instrument. A measurement instrument gives us readings that we call observed values. The reliability $\Delta$ of an instrument is defined as the ratio of the true over the observed measure. Under the parallel model, one can show that the reliability of any variable $X_j$ (as an instrument to measure the true value) is given by

$$\rho = \sigma_j/(\sigma_j^2 + \sigma^2)$$

which is also the constant correlation between any two variables. This coefficient is also known as the intraclass coefficient. The reliability coefficient $\Delta$ can be easily interpreted as a correlation coefficient between the true and the observed measure.

When the parallel model is assumed, the reliability of the sum of $k$ variables equals

$$\tilde{\rho} = kp/[kp + (1 - \rho)]$$

This formula is known as the Spearman-Brown formula. Its maximum-likelihood estimator, under the assumption of a normal distribution of the error and the parallel model, is known as the Cronbach $\alpha$ coefficient (CAC):

$$\alpha = [k/(k - 1)] \left[1 - \left(\sum_{j=1}^{n} S_j^2/S_{tot}^2\right)\right],$$

where

$$S_j^2 = 1/(n - 1) \sum_{i=1}^{n} (X_{ij} - \bar{X}_j)^2$$

and

$$S_{tot}^2 = 1/(nk - 1) \sum_{j=1}^{k} \sum_{i=1}^{n} (X_{ij} - \bar{X})^2.$$


15. Kristof W. The statistical theory of stepped-up reliability coefficients when a test has been divided into several equivalent parts. *Psychometrika.* 1963;28:221–238.